INCOMPRESSIBLE LIMIT FOR THE 2D ISENTROPIC EULER SYSTEM WITH CRITICAL INITIAL DATA

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ABSTRACT. We study in this paper the low Mach number limit for the $2d$ isentropic Euler system with ill-prepared initial data belonging to the critical Besov space $B_{2,1}^2$. By combining Strichartz estimates with the special structure of the vorticity we prove that the lifespan goes to infinity as the Mach number goes to zero. We also prove the strong convergence in the space of the initial data $B_{2,1}^2$ of the incompressible parts to the solution of the incompressible Euler system. There are at least two main difficulties: the first one concerns the Beale-Kato-Majda criterion which is not known to work for rough regularities. However, the second one is related to the critical aspect of the Strichartz norm $\|(\operatorname{div} v_\varepsilon, \nabla c_\varepsilon)\|_{L_T^4 L^\infty}$ which has the scale of $B_{2,1}^2$ in the space variable.

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1. INTRODUCTION

This work is devoted to the study of the isentropic Euler equations describing the evolution of a compressible fluid evolving in the whole space and under a constant entropy constraint. The system is given by the coupled equations

$$
(1) \qquad
\begin{cases}
\rho(\partial_t u + u \cdot \nabla u) + \nabla \rho^\gamma = 0, \ \gamma > 1 \\
\partial_t \rho + \operatorname{div}(\rho\, u) = 0 \\
(u, \rho)_{|t=0} = (u_0, \rho_0).
\end{cases}
$$

Here, u stands for the velocity field which is a time-dependent vector field over $\mathbb{R}^d$, $\rho > 0$ is the density which is assumed to be nonnegative in order to avoid the vacuum. The parameter $\gamma > 1$ is the adiabatic exponent. The local well-posedness theory can be achieved by using the

symmetrization of Kawashima Makino and Ukai [20]. In other words, we introduce the sound speed defined by,

$$c = 2\frac{\sqrt{\gamma}}{\gamma - 1}\rho^{\frac{\gamma-1}{2}}$$

and thus the system (1) reduces to

(2)
$$\begin{cases} \partial_t u + u \cdot \nabla u + \bar{\gamma} c \nabla c = 0 \\ \partial_t c + u \cdot \nabla c + \bar{\gamma} c \operatorname{div} u = 0 \\ (u, c)_{|t=0} = (u_0, c_0), \end{cases}$$

with $\bar{\gamma} := \frac{\gamma-1}{2}$. This form fits with the theory of the symmetric quasilinear hyperbolic systems developed by Klainerman and Majda and which answers the local well-posedness in the framework of Sobolev spaces $H^s, s > \frac{d}{2} + 1$, see [15, 16]. It is well-known that the system (2) can be considered as a good approximation of the incompressible Euler equations, despite we make some suitable assumptions as we will see later. The first step to reach this result is to introduce a suitable rescaling and change of variables. This can be done by making a perturbation of order ε from the constant background state $(0, c_0)$:

$$u(t, x) = \bar{\gamma}c_0\varepsilon v_\varepsilon(\varepsilon\bar{\gamma}c_0 t, x), \ c(t, x) = c_0 + \bar{\gamma}c_0\varepsilon c_\varepsilon(\varepsilon\bar{\gamma}c_0 t, x).$$

Therefore we get the system

(3)
$$\begin{cases} \partial_t v_\varepsilon + v_\varepsilon \cdot \nabla v_\varepsilon + \bar{\gamma}c_\varepsilon\nabla c_\varepsilon + \frac{1}{\varepsilon}\nabla c_\varepsilon = 0 \\ \partial_t c_\varepsilon + v_\varepsilon \cdot \nabla c_\varepsilon + \bar{\gamma}c_\varepsilon\operatorname{div} v_\varepsilon + \frac{1}{\varepsilon}\operatorname{div} v_\varepsilon = 0 \\ (v_\varepsilon, c_\varepsilon)_{|t=0} = (v_{0,\varepsilon}, c_{0,\varepsilon}). \end{cases}$$

The parameter ε is called the Mach number and measures the compressibility of the fluid. For more details about the derivation of this last model we refer for instance to [12, 15, 21]. The mathematical study of this model is well developed and there are a lot of papers devoted to the rigorous justification of the convergence towards the incompressible Euler equations when the Mach number goes to zero. We recall that the incompressible Euler system is given by,

(4)
$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = 0 \\ \operatorname{div} v = 0 \\ v|_{t=0} = v_0. \end{cases}$$

We point out that the issue to this problem depends on several factors like the geometry of the domain where the fluid is assumed to evolve or the state of the initial data. For example, the analysis in the case of the full space is completely different from the torus due to the resonance phenomenon. The second factor is related to the structure of the initial data and whether they are well-prepared or not. In the well-prepared case, the initial data are assumed to be slightly compressible, which means that $(\operatorname{div} v_{0,\varepsilon}, \nabla c_{0,\varepsilon}) = O(\varepsilon)$ as $\varepsilon \to 0$. This assumption allows to get a uniform bound for $(\partial_t v_\varepsilon)_\varepsilon$, and consequently we can pass to the limit by using Aubin-Lions compactness lemma, for more details see [15, 16]. However, in the ill-prepared case, the family $(v_{0,\varepsilon}, c_{0,\varepsilon})_\varepsilon$ is only assumed to be bounded in some Sobolev spaces H^s with $s > \frac{d}{2} + 1$ and the incompressible parts of $(v_{0,\varepsilon})_\varepsilon$ converge to some vector field v_0. This case is more subtle because the time derivative $\partial_t v_\varepsilon$, is of size $O(\frac{1}{\varepsilon})$ and the preceding compactness method is out of use. To overcome this difficulty, Ukai [27] used the dispersive effects generated by the acoustic waves in order to prove that the compressible part of the velocity and the acoustic term vanish when ε goes to zero. We point out that this problem has already been studied in numerous papers, see for instance [1, 3, 8, 10, 11, 15, 16, 17, 18, 19, 21, 24, 27].

Concerning the blow up theory, it is well-known that contrary to the incompressible Euler system, the equations (1) develop in space dimension two singularities in finite time and for some smooth initial data, see [22]. This phenomenon holds true for higher dimensions, see [25]. On the other

hand, it was shown in [16] that for the whole space or for the torus domain when the limit system (4) exists for some time interval $[0, T_0]$ then for ever $T < T_0$ there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ the solution to (3) exists till the time T. This result was established for the well-prepared case and with sufficient smooth initial data, that is, in H^s and $s > \frac{d}{2} + 2$. The proof relies on the perturbation theory which ceases to work for lower regularities $\frac{d}{2} + 2 > s > \frac{d}{2} + 1$. Consequently, if we denote by T_ε the lifespan of the solution $(v_\varepsilon, c_\varepsilon)$ then we get in space dimension two that $\lim_{\varepsilon \to 0} T_\varepsilon = +\infty$. It seems that we can get better information about T_ε when the initial data have some special structures. In [2], Alinhac proved that in space dimension two and for axisymmetric initial data the lifespan of the solution is equivalent to $\frac{1}{\varepsilon}$. In dimension three and for irrotational velocity, Sideris [25] established that the solutions are almost global in time, that is, their lifespan are bounded below by $e^{\frac{1}{\varepsilon}}$. Finally, we mention the results of [9, 23] dealing with the global existence under some suitable conditions on the initial data: the initial density must be small and has a compact support and the spectrum of ∇u_0 must be far away the set of the negative real numbers. We intend in this paper to deal with the incompressible limit problem in space dimension two but for the critical Besov regularity $B_{2,1}^2$. We emphasize that by using Strichartz estimates we can prove that for initial data with sub-critical regularity, that is H^s and $s > 2$, the lifespan is bounded below by $C \log \log \frac{1}{\varepsilon}$. We can interpret this double logarithmic growth by the fact that the Sobolev norm of the solutions to the incompressible Euler equations has at most a double exponential growth in time. Our main result reads as follows.

Theorem 1. *Let $\{(v_{0,\varepsilon}, c_{0,\varepsilon})_{0 < \varepsilon \leq 1}\}$ be a family of initial data such that*

$$\sum_{q \geq -1} 2^{2q} \sup_{0 < \varepsilon \leq 1} \|(\Delta_q v_{0,\varepsilon}, \Delta_q c_{0,\varepsilon})\|_{L^2} < +\infty.$$

Then the system (3) has a unique solution $(v_\varepsilon, c_\varepsilon) \in \mathcal{C}([0, T_\varepsilon]; B_{2,1}^2)$ with

$$\lim_{\varepsilon \to 0} T_\varepsilon = +\infty.$$

Moreover, the acoustic parts of the solutions tend to zero:

$$\lim_{\varepsilon \to 0} \|(\operatorname{div} v_\varepsilon, \nabla c_\varepsilon)\|_{L_{T_\varepsilon}^1 L^\infty} = 0.$$

Assume in addition that $\lim_{\varepsilon \to 0} \|\mathbb{P} v_{0,\varepsilon} - v_0\|_{L^2} = 0$, for some $v_0 \in B_{2,1}^2$. Then the incompressible parts $(\mathbb{P} v_\varepsilon)_\varepsilon$ converge to the solution v of the system (4) associated to the initial data v_0. More precisely, for every $T > 0$,

$$\lim_{\varepsilon \to 0} \|\mathbb{P} v_\varepsilon - v\|_{L_T^\infty B_{2,1}^2} = 0.$$

Where we denote by $\mathbb{P} v = v - \nabla \Delta^{-1} \operatorname{div} v$, the Leray's projector over solenoidal vector fields.

Before discussing the important ingredients of the proof, we will give some useful remarks.

Remark 1.   (1) The result of Theorem 1 will be generalized in Theorem 3. This latter one gives a precise information about the lifespan: we get

$$T_\varepsilon \geq C_0 \log \log \Psi(\log \frac{1}{\varepsilon}),$$

where Ψ is an implicit function depending on the profile of the initial data.

(2) In the second part of Theorem 1, the velocity v_0 belongs naturally to the space L^2, but according to the assumption $\sum_{q \geq -1} 2^{2q} \sup_{0 < \varepsilon \leq 1} \|\Delta_q v_{0,\varepsilon}\|_{L^2} < +\infty$, combined with the weak compactness of the Besov space $B_{2,1}^2$, we obtain $v_0 \in B_{2,1}^2$. Moreover we get the strong convergence to v_0 in the space $B_{2,1}^2$. Indeed, let $N \in \mathbb{N}^*$, then we can write

$$\|\mathbb{P} v_{0,\varepsilon} - v_0\|_{B_{2,1}^2} \lesssim 2^{2N} \|\mathbb{P} v_{0,\varepsilon} - v_0\|_{L^2} + \sum_{q \geq N} 2^{2q} \big(\|\Delta_q v_0\|_{L^2} + \sup_{0 < \varepsilon \leq 1} \|\Delta_q v_{0,\varepsilon}\|_{L^2}\big).$$

Thus, letting ε go to zero, we infer

$$\limsup_{\varepsilon \to 0} \|\mathbb{P}v_{0,\varepsilon} - v_0\|_{B^2_{2,1}} \lesssim \sum_{q \geq N} 2^{2q} \big(\|\Delta_q v_0\|_{L^2} + \sup_{0 < \varepsilon \leq 1} \|\Delta_q v_{0,\varepsilon}\|_{L^2} \big)$$

and therefore when N goes to $+\infty$ yields the desired result.

Let us now describe the main difficulties that emerge when we try to work in the critical framework and give the basic ideas to solve them. We distinguish two principal difficulties: The first one has a compressible nature and concerns the estimate of the acoustic parts $\|(\mathrm{div}\, v_{\varepsilon}, \nabla c_{\varepsilon})\|_{L^1_T L^\infty}$. Since in the space variable the norm of the acoustic parts scales like the space of the initial data, it seems then so hard to get an explicit rate on ε for these quantities. It is worthy pointing out that for the sub-critical regularities the rate convergence to zero is of order ε^θ, for some $\theta > 0$. To overcome this critical aspect, we will proceed in the spirit of the the paper [14]: we can find an adequate nondecreasing profile Ψ such that $\lim_{q \to +\infty} \Psi(q) = +\infty$ and:

$$\sup_{\varepsilon} \sum_q \Psi(q) 2^{2q} \|(\Delta_q v_{0,\varepsilon}, \Delta_q c_{0,\varepsilon})\|_{L^2} < +\infty.$$

Then by performing energy estimate we prove that the solutions have the same additional decay in the time interval $[0, T_\varepsilon]$. This means that there is no energy transfer to higher frequencies. Consequently, interpolating between this estimate and Strichartz estimate for the lower degree quantity $\|(\nabla \Delta^{-1} v_\varepsilon, c_\varepsilon)\|_{L^1_T L^\infty}$, we get that the acoustic parts vanish when the Mach number goes to zero. However the rate convergence is implicit and can be related to the distribution on Fourier modes of the energy of the initial data.

The second difficulty that one should to deal with has an incompressible nature and concerns the Beale-Kato-Majda criterion [4]. It was initially stated for the system (4) and for the sub-critical spaces $H^s, s > \frac{d}{2}+1$. It asserts that we can propagate the initial regularity in the time interval $[0, T]$ provided that we can control the quantity $\|\omega\|_{L^1_T L^\infty}$, where ω denotes the vorticity of the velocity. As a matter of fact, we get in space dimension two the global existence of the strong solutions since the vorticity is only transported by the flow and thus its L^∞ norm is conserved. Unfortunately, this criterion is not known to work in the critical Besov space $B^{\frac{d}{2}+1}_{2,1}$ and it is replaced by a stronger norm $\|\omega\|_{L^1_T B^0_{\infty,1}}$. The global existence for $2d$ case was solved some years ago by Vishik [28] who was able to prove the following linear growth:

$$\|\omega(t)\|_{B^0_{\infty,1}} \leq C \|\omega_0\|_{B^0_{\infty,1}} \Big(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \Big).$$

The Vishik's method relies on a composition law in Besov space $B^0_{\infty,1}$ with a logarithmic growth. We emphasize that in [13], we gave an other proof which has the advantage to fit with general models like transport-diffusion equations. Now, in order to get a lower bound for the lifespan satisfying $\lim_{\varepsilon \to 0} T_\varepsilon = +\infty$ we should establish an analogous estimate in the compressible case. This is not an easy task due to the nonlinearities in the vorticity equation and to the lack of the incompressibility of the velocity. We recall that the vorticity $\omega_\varepsilon = \partial_1 v^2_\varepsilon - \partial_2 v^1_\varepsilon$ satisfies the compressible transport equation:

$$\partial_t \omega_\varepsilon + v_\varepsilon \cdot \nabla \omega_\varepsilon + \omega_\varepsilon \mathrm{div}\, v_\varepsilon = 0.$$

We will prove in Theorem 2 the following key estimate: for $p \in [1, +\infty[$

$$\|\omega_\varepsilon(t)\|_{B^0_{\infty,1}} \leq C \|\omega_{0,\varepsilon}\|_{B^0_{\infty,1}} \Big(1 + e^{C\|\nabla v_\varepsilon\|_{L^1_t L^\infty}} \|\mathrm{div}\, v_\varepsilon\|^2_{L^1_t B^{\frac{2}{p}}_{p,1}} \Big) \Big(1 + \int_0^t \|\nabla v_\varepsilon(\tau)\|_{L^\infty} d\tau \Big).$$

We observe that for $\mathrm{div}\, v_\varepsilon = 0$ we get the Vishik's estimate. The proof will be done in the spirit of the work [13] but with slight modifications. In the first step, we use the Lagrangian coordinates

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combined with a filtration procedure to get rid of the compressible part. In the second step we establish a suitable splitting of the vorticity and use a dynamical interpolation method.

Concerning the incompressible limit we prove first the strong convergence in $L^\infty_{\text{loc}}(\mathbb{R}_+; L^2)$ by using Strichartz estimates. This leads according to some standard interpolation inequalities to the strong convergence in Besov spaces $B^s_{2,1}$, for all $s < 2$. However for the strong convergence in the space of the initial data $B^2_{2,1}$, we establish an additional frequency decay of the energy uniformly with respect to t and ε. In other words, from the assumption on the initial data we can find a nondecreasing function Ψ independent of ε such that $(v_{0,\varepsilon}, c_{0,\varepsilon})$ belongs to $B^{2,\Psi}_{2,1}$, see the Definition 1. This regularity will be conserved in the full interval $[0, T_\varepsilon]$ and thus the energy transfer for higher frequencies can not occur.

The paper is organized as follows: we recall in section 2 some basic results about the Littlewood-Paley operators and Besov spaces. In section 3, we gather some energy estimates for the heterogeneous Besov spaces $B^{s,\Psi}_{2,1}$ and we establish some useful Strichartz estimates for the acoustic parts of the fluid. In section 4, we establish a logarithmic estimate for a compressible transport model. In section 5, we generalize the result of Theorem 1 and we give the proofs.

2. Basic Tools

In this preliminary section, we are going to recall the Littlewood-Paley operators and give some of their elementary properties. We will also introduce some function spaces and review some important lemmas that will be used later.

2.1. Littlewood-Paley operators.

We denote by C any positive constant that may change from line to line and C_0 a real positive constant depending on the size of the initial data. We will use the following notations: for any positive A and B, the notation $A \lesssim B$ means that there exists a positive constant C independent on A and B and such that $A \leqslant CB$.

To define Besov spaces we need to the following dyadic unity partition (see [7]). There are two nonnegative radial functions $\chi \in \mathcal{D}(\mathbb{R}^2)$ and $\varphi \in \mathcal{D}(\mathbb{R}^2 \backslash \{0\})$ such that

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^2,$$

$$\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^2 \backslash \{0\},$$

$$|p - q| \geq 2 \Rightarrow \text{supp } \varphi(2^{-p}\cdot) \cap \text{supp } \varphi(2^{-q}\cdot) = \varnothing,$$

$$q \geq 1 \Rightarrow \text{supp } \chi \cap \text{supp } \varphi(2^{-q}\cdot) = \varnothing.$$

Let $u \in \mathcal{S}'(\mathbb{R}^2)$ we define the Littlewood-Paley operators by

$$\Delta_{-1}u = \chi(\text{D})u, \ \ \forall q \geq 0, \ \Delta_q u = \varphi(2^{-q}\text{D})u \ \text{ and } \ S_q u = \sum_{-1 \leq p \leq q-1} \Delta_p u.$$

We can easily check that

$$u = \sum_{q \in \mathbb{Z}} \Delta_q u, \ \ \forall u \in \mathcal{S}'(\mathbb{R}^2).$$

Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality: for any $u, v \in \mathcal{S}'(\mathbb{R}^2)$,

$$\Delta_p \Delta_q u = 0 \quad \text{ if } \quad |p - q| \geqslant 2$$

$$\Delta_p(S_{q-1}u \Delta_q v) = 0 \quad \text{ if } \quad |p - q| \geqslant 5.$$

Let us note that the above operators Δ_q and S_q map continuously L^p into itself uniformly with respect to q and p. By the same way we define the homogeneous operators:

$$\forall q \in \mathbb{Z}, \quad \dot{\Delta}_q u = \varphi(2^{-q}\mathrm{D})v \quad \text{and} \quad \dot{S}_q u = \sum_{j \leq q-1} \dot{\Delta}_j u.$$

We notice that these operators are of convolution type. For example for $q \in \mathbb{Z}$, we have

$$\dot{\Delta}_q u = 2^{2q}h(2^q \cdot) * u, \quad \text{with} \quad h \in \mathcal{S}, \quad \widehat{h}(\xi) = \varphi(\xi).$$

Now we recall Bernstein inequalities, see for exapmle [7].

Lemma 1. *There exists a constant $C > 0$ such that for all $q \in \mathbb{N}$, $k \in \mathbb{N}$ and for every tempered distriubution u we have*

$$\sup_{|\alpha|=k} \|\partial^\alpha S_q u\|_{L^b} \leqslant C^k 2^{q\left(k+2\left(\frac{1}{a}-\frac{1}{b}\right)\right)} \|S_q u\|_{L^a} \quad \text{for} \quad b \geqslant a \geqslant 1$$

$$C^{-k} 2^{qk} \|\Delta_q u\|_{L^a} \leqslant \sup_{|\alpha|=k} \|\partial^\alpha \Delta_q u\|_{L^a} \leqslant C^k 2^{qk} \|\Delta_q u\|_{L^a}.$$

2.2. **Besov spaces.** Now we will define the Besov spaces by using Littlewood-Paley operators. Let $(p,r) \in [1,+\infty]^2$ and $s \in \mathbb{R}$, then the non-homogeneous Besov space $B^s_{p,r}$ is the set of tempered distributions u such that

$$\|u\|_{B^s_{p,r}} := \left(2^{qs} \|\Delta_q u\|_{L^p}\right)_{\ell^r} < +\infty.$$

The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as the set of $u \in \mathcal{S}'(\mathbb{R}^d)$ up to polynomials such that

$$\|u\|_{\dot{B}^s_{p,r}} := \left(2^{qs} \|\dot{\Delta}_q u\|_{L^p}\right)_{\ell^r(\mathbb{Z})} < +\infty.$$

We remark that the usual Sobolev space H^s coincides with $B^s_{2,2}$ for $s \in \mathbb{R}$ and the Hölder space C^s coincides with $B^s_{\infty,\infty}$ when s is not an integer.
The following embeddings are an easy consequence of Bernstein inequalities,

$$B^s_{p_1,r_1} \hookrightarrow B^{s+2\left(\frac{1}{p_2}-\frac{1}{p_1}\right)}_{p_2,r_2}, \qquad p_1 \leq p_2 \quad and \quad r_1 \leq r_2.$$

Let $T > 0$ and $\rho \geq 1$, we denote by $L^\rho_T B^s_{p,r}$ the space of tempered distributions u such that

$$\|u\|_{L^\rho_T B^s_{p,r}} := \left\|\left(2^{qs} \|\Delta_q u\|_{L^p}\right)_{\ell^r}\right\|_{L^\rho_T} < +\infty.$$

Now we will introduce the heterogeneous Besov spaces which are an extension of the classical Besov spaces.

Definition 1. Let $\Psi : \{-1\} \cup \mathbb{N} \to \mathbb{R}^*_+$ be a given function.
(i) We say that Ψ belongs to the class $\mathcal{U}$ if the following conditions are satisfied:
   (1) Ψ is a nondecreasing function.
   (2) There exists $C > 0$ such that

$$\sup_{x \in \mathbb{N}\cup\{-1\}} \frac{\Psi(x+1)}{\Psi(x)} \leq C.$$

(ii) We define the class $\mathcal{U}_\infty$ by the set of function $\Psi \in \mathcal{U}$ satisfying $\lim_{x \to +\infty} \Psi(x) = +\infty$.

(iii) Let $s \in \mathbb{R}, (p,r) \in [1,+\infty]^2$ and $\Psi \in \mathcal{U}$. We define the heterogeneous Besov space $B^{s,\Psi}_{p,r}$ as follows:

$$u \in B^{s,\Psi}_{p,r} \iff \|u\|_{B^{s,\Psi}_{p,r}} = \left(\Psi(q)2^{qs} \|\Delta_q u\|_{L^p}\right)_{\ell^r} < +\infty.$$

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Remark 2.   (1) From the part (2) of the above Definition, we see that the profile Ψ has at most an exponential growth: there exists $\alpha > 0$ such that

$$\Psi(-1) \le \Psi(q) \le \Psi(-1)e^{\alpha q}, \quad \forall q \ge -1.$$

When the profile Ψ has an exponential growth: $\Psi(q) = 2^{\alpha q}$ with $\alpha \in \mathbb{R}_+$, the space $B_{p,r}^{s,\Psi}$ reduces to the classical Besov space $B_{p,r}^{s+\alpha}$.

(2) The condition (2) seems to be necessary for the definition of $B_{p,r}^{s,\Psi}$: it allows to get a coherent definition which is independent of the choice of the dyadic partition.

The following lemma proved in [14] is important for the proof of Theorem 1. Roughly speaking, it says that any element of a given Besov space is always more regular than its prescribed regularity.

Lemma 2. *Let* $s \in \mathbb{R}$, $p \in [1, +\infty]$, $r \in [1, +\infty[$ *and* $f \in B_{p,r}^s$. *Then there exists a function* $\Psi \in \mathcal{U}_\infty$ *such that* $f \in B_{p,r}^{s,\Psi}$.

This yields to the following result, see [14].

Corollary 1. *Let* $s \in \mathbb{R}, p \in [1, +\infty], r \in [1, +\infty[$ *and* $(g_\varepsilon)_{0 < \varepsilon \le 1}$ *be a family of smooth functions satisfying*

$$\left(\sum_{q \ge -1} 2^{qsr} \sup_{0 < \varepsilon \le 1} \|\Delta_q g_\varepsilon\|_{L^p}^r \right)^{\frac{1}{r}} < +\infty.$$

Then there exists $\Psi \in \mathcal{U}_\infty$ *such that* $g_\varepsilon \in B_{p,r}^{s,\Psi}$, *with uniform bounds, that is,*

$$\sup_{0 < \varepsilon \le 1} \left(\sum_{q \ge -1} \Psi^r(q) 2^{qsr} \|\Delta_q g_\varepsilon\|_{L^p}^r \right)^{\frac{1}{r}} < +\infty.$$

For the proof of the next proposition, see for example [12].

Lemma 3. *Let* u *be a smooth vector field, not necessary of zero divergence. Let* f *be a smooth solution of the transport equation*

$$\partial_t f + u \cdot \nabla f = g, \ f_{|t=0} = f_0,$$

such that $f_0 \in B_{p,r}^s(\mathbb{R}^2)$ *and* $g \in L_{\text{loc}}^1(\mathbb{R}_+; B_{p,r}^s)$. *Then the following assertions hold true*

(1) *Let* $(p,r) \in [1, \infty]^2$ *and* $s \in]0, 1[$, *then*

$$\|f(t)\|_{B_{p,r}^s} \le Ce^{CV(t)} \left(\|f_0\|_{B_{p,r}^s} + \int_0^t e^{-CV(\tau)} \|g(\tau)\|_{B_{p,r}^s} d\tau \right),$$

where $V(t) = \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau$ *and* C *is a constant depending on* s.

(2) *Let* $s \in]-1, 0]$, $r \in [1, +\infty]$ *and* $p \in [2, +\infty]$ *with* $s + \frac{2}{p} > 0$, *then*

$$\|f(t)\|_{B_{p,r}^s} \le Ce^{CV_p(t)} \left(\|f_0\|_{B_{p,r}^s} + \int_0^t e^{-CV_p(\tau)} \|g(\tau)\|_{B_{p,r}^s} d\tau \right),$$

with $V_p(t) = \|\nabla u\|_{L_t^1 L^\infty} + \|\text{div} u\|_{L_t^1 B_{p,\infty}^{\frac{2}{p}}}$.

Now, we prove the following result which will be needed later.

Lemma 4. *Let* v *be a vector field such that* $v \in B_{\infty,1}^1 \cap L^2$ *and* ω *its vorticity. Then we have*

$$\|\nabla v\|_{L^\infty} \lesssim \|v\|_{L^2} + \|\text{div} \, v\|_{B_{\infty,1}^0} + \|\omega\|_{B_{\infty,1}^0}.$$

Proof. We split the velocity into incompressible and compressible parts: $v = \mathbb{P}v + \mathbb{Q}v$. Then we have

$$\operatorname{curl} v = \operatorname{curl} \mathbb{P} v$$

We have by Bernstein inequality, the continuity of $\dot{\Delta}_q \mathbb{P} : L^p \to L^p$, $\forall p \in [1, \infty]$ uniformly in q and $\|\dot{\Delta}_q v\|_{L^\infty} \sim 2^{-q} \|\dot{\Delta}_q \omega\|_{L^\infty}$, that

$$
\begin{aligned}
\|\nabla \mathbb{P} v\|_{L^\infty} &\leq \|\dot{\Delta}_{-1} \nabla \mathbb{P} v\|_{L^\infty} + \sum_{q \in \mathbb{N}} \|\dot{\Delta}_q \nabla \mathbb{P} v\|_{L^\infty} \\
&\lesssim \|\dot{\Delta}_{-1} \mathbb{P} v\|_{L^2} + \sum_{q \in \mathbb{N}} 2^q \|\dot{\Delta}_q \mathbb{P} v\|_{L^\infty} \\
&\lesssim \|v\|_{L^2} + \sum_{q \in \mathbb{N}} \|\Delta_q \omega\|_{L^\infty} \\
&\lesssim \|v\|_{L^2} + \|\omega\|_{B^0_{\infty,1}}.
\end{aligned}
$$

(5)

On the other hand, using Bernstein inequality leads to

$$
\begin{aligned}
\|\nabla \mathbb{Q} v\|_{L^\infty} &= \|\nabla^2 \Delta^{-1} \operatorname{div} v\|_{L^\infty} \\
&\leq \|\Delta_{-1} \nabla^2 \Delta^{-1} \operatorname{div} v\|_{L^\infty} + \sum_{q \geq 0} \|\Delta_q \nabla^2 \Delta^{-1} \operatorname{div} v\|_{L^\infty} \\
&\lesssim \|v\|_{L^2} + \|\operatorname{div} v\|_{B^0_{\infty,1}}.
\end{aligned}
$$

Combining this estimate with (5), gives

$$\|\nabla v\|_{L^\infty} \lesssim \|v\|_{L^2} + \|\operatorname{div} v\|_{B^0_{\infty,1}} + \|\omega\|_{B^0_{\infty,1}}.$$

This achieves the proof of the lemma. $\square$

3. Preliminaries

This section is devoted to some useful estimates for the system (3) that will be crucial for the proof of the main results. We will discuss some energy estimates for the full system (3) and give some Strichartz estimates for the acoustic operator.

3.1. **Energy estimates.** Here we list two energy estimates for (3). The first one is very classical and is concerned with the L^2-estimate. However the second one deals with the energy estimates in the heterogeneous Besov space $B^{s,\Psi}_{2,1}$. All the spaces of the initial data are constructed over the Lebesgue space L^2 in order to remove the singular terms and get uniform estimates with respect to the Mach number ε. For the proof, see for instance [14].

Proposition 1. *Let $(v_\varepsilon, c_\varepsilon)$ be a smooth solution of (3) and $\Psi \in \mathcal{U}$, see the Definition 1.*

(1) L^2-*estimate: there exists $C > 0$ such that $\forall t \geq 0$*

$$\|(v_\varepsilon, c_\varepsilon)(t)\|_{L^2} \leq C \|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{L^2} e^{C \|\operatorname{div} v_\varepsilon\|_{L^1_t L^\infty}}.$$

(2) *Besov estimates: for $s > 0$ and $r \in [1, +\infty]$, there exists $C > 0$ such that*

$$\|(v_\varepsilon, c_\varepsilon)(t)\|_{B^{s,\Psi}_{2,r}} \leq C \|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B^{s,\Psi}_{2,r}} e^{C V_\varepsilon(t)}$$

with

$$V_\varepsilon(t) := \|\nabla v_\varepsilon\|_{L^1_t L^\infty} + \|\nabla c_\varepsilon\|_{L^1_t L^\infty}.$$

We emphasize that for $\Psi \equiv 1$ the space $B_{2,r}^{s,\Psi}$ reduces to the classical Besov space $B_{2,r}^s$. Therefore we get the known energy estimate, see for instance [12],

$$(6) \qquad \|(v_\varepsilon, c_\varepsilon)(t)\|_{B_{2,r}^s} \leq C\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B_{2,r}^s}\, e^{CV_\varepsilon(t)}.$$

3.2. **Strichartz estimates.** Our goal is to establish some classical Strichartz estimates for the acoustic parts which are governed by wave equations. Roughly speaking, we will show that the averaging in time of the compressible part of the velocity v_ε and the sound speed c_ε will lead to vanishing quantities when ε goes to zero. We point out that Ukai [27] used this fact but under its dispersion form in order to study the incompressible limit in the framework of the ill-prepared initial data. There are various ways to exhibit the wave structure and we prefer here the complex version. We start with rewriting the system (3) under the form:

$$(7) \qquad \begin{cases} \partial_t v_\varepsilon + \frac{1}{\varepsilon}\nabla c_\varepsilon = -v_\varepsilon \cdot \nabla v_\varepsilon - \bar{\gamma}c_\varepsilon \nabla c_\varepsilon := f_\varepsilon \\ \partial_t c_\varepsilon + \frac{1}{\varepsilon}\mathrm{div}\, v_\varepsilon = -v_\varepsilon \cdot \nabla c_\varepsilon - \bar{\gamma}c_\varepsilon \mathrm{div}\, v_\varepsilon := g_\varepsilon \\ (v_\varepsilon, c_\varepsilon)_{|t=0} = (v_{0,\varepsilon}, c_{0,\varepsilon}). \end{cases}$$

Denote by $\mathbb{Q}v := \nabla\Delta^{-1}\mathrm{div}\, v$ the compressible part of the velocity v. Then the quantity

$$\Gamma_\varepsilon := \mathbb{Q}v_\varepsilon - i\nabla|\mathrm{D}|^{-1}c_\varepsilon, \qquad \text{with } |\mathrm{D}| = (-\Delta)^{\frac{1}{2}}$$

satisfies the following wave equation

$$(8) \qquad (\partial_t + \frac{i}{\varepsilon}|\mathrm{D}|)\Gamma_\varepsilon = \mathbb{Q}f_\varepsilon - i\nabla|\mathrm{D}|^{-1}g_\varepsilon.$$

Similarly, we can easily check that the quantity

$$\Upsilon_\varepsilon := |\mathrm{D}|^{-1}\mathrm{div}\, v_\varepsilon + ic_\varepsilon$$

obeys to the wave equation

$$(9) \qquad (\partial_t + \frac{i}{\varepsilon}|\mathrm{D}|)\Upsilon_\varepsilon = |\mathrm{D}|^{-1}\mathrm{div}\, f_\varepsilon + ig_\varepsilon.$$

Now, we are in a position to use of the following Strichartz estimates stated in space dimension two and for the proof see for instance [6, 12, 26].

Lemma 5. *Let ψ be a solution of the wave equation*

$$(\partial_t + \frac{i}{\varepsilon}|\mathrm{D}|)\psi = G, \qquad \psi_{|t=0} = \psi_0.$$

Then there exists an absolute constant C such that for every $T > 0$,

$$\|\psi\|_{L_T^r L^p} \leq C\varepsilon^{\frac{1}{4}-\frac{1}{2p}}\left(\|\psi_0\|_{\dot{B}_{2,1}^{\frac{3}{4}-\frac{3}{2p}}} + \|G\|_{L_T^1 \dot{B}_{2,1}^{\frac{3}{4}-\frac{3}{2p}}}\right),$$

for all $p \in [2, +\infty]$ and $r = 4 + \frac{8}{p-2}$.

This result enables us to get the following estimates.

Proposition 2. *Let $(v_{0,\varepsilon}, c_{0,\varepsilon})$ be a bounded family in $B_{2,1}^{\frac{7}{4}}$. Then any smooth solution of (3) defined in the time interval $[0,T]$ satisfies,*

$$\|(\mathbb{Q}v_\varepsilon, c_\varepsilon)\|_{L_T^4 L^\infty} \leq C_0 \varepsilon^{\frac{1}{4}}(1+T)e^{CV_\varepsilon(T)},$$

where C_0 depends on the quantities $\sup_{0<\varepsilon\leq 1}\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B_{2,1}^{\frac{7}{4}}}$ and

$$V_\varepsilon(T) := \|\nabla v_\varepsilon\|_{L_T^1 L^\infty} + \|\nabla c_\varepsilon\|_{L_T^1 L^\infty}.$$

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Proof. We apply Lemma 5 to the equation (8) with $p = +\infty$, then

$$\|\Gamma_\varepsilon\|_{L_T^4 L^\infty} \lesssim \varepsilon^{\frac{1}{4}} \big(\|\Gamma_\varepsilon^0\|_{\dot{B}_{2,1}^{\frac{3}{4}}} + \|\mathbb{Q}f_\varepsilon - i\nabla|\mathrm{D}|^{-1}g_\varepsilon\|_{L_T^1 \dot{B}_{2,1}^{\frac{3}{4}}} \big).$$

Using the continuity of the Riesz operators $\mathbb{Q}$ and $\nabla|\mathrm{D}|^{-1}$ on the homogeneous Besov spaces, combined with the embedding $B_{2,1}^{\frac{3}{4}} \hookrightarrow \dot{B}_{2,1}^{\frac{3}{4}}$ and Hölder inequality

$$\|\Gamma_\varepsilon\|_{L_T^4 L^\infty} \lesssim \varepsilon^{\frac{1}{4}} \big(\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{\dot{B}_{2,1}^{\frac{3}{4}}} + \|(f_\varepsilon, g_\varepsilon)\|_{L_T^1 \dot{B}_{2,1}^{\frac{3}{4}}} \big)$$

(10)
$$\lesssim \varepsilon^{\frac{1}{4}} \big(\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B_{2,1}^{\frac{3}{4}}} + T\|(f_\varepsilon, g_\varepsilon)\|_{L_T^\infty B_{2,1}^{\frac{3}{4}}} \big).$$

It remains to estimate $\|(f_\varepsilon, g_\varepsilon)\|_{L_T^\infty B_{2,1}^{\frac{3}{4}}}$. For this purpose we use the following law product which can be proved for example by using Bony's decomposition :

$$\|f \cdot \nabla g\|_{B_{2,1}^{\frac{3}{4}}} \lesssim \|f\|_{L^\infty}\|g\|_{B_{2,1}^{\frac{7}{4}}} + \|g\|_{L^\infty}\|f\|_{B_{2,1}^{\frac{7}{4}}}.$$

Thus we get

$$\|(f_\varepsilon, g_\varepsilon)\|_{B_{2,1}^{\frac{3}{4}}} \lesssim \|(v_\varepsilon, c_\varepsilon)\|_{L^\infty}\|(v_\varepsilon, c_\varepsilon)\|_{B_{2,1}^{\frac{7}{4}}}$$
$$\lesssim \|(v_\varepsilon, c_\varepsilon)\|_{B_{2,1}^{\frac{7}{4}}}^2,$$

where we have used in the last line the embedding $B_{2,1}^{\frac{7}{4}} \hookrightarrow L^\infty$. Therefore (6) yields to

$$\|(f_\varepsilon, g_\varepsilon)\|_{L_T^\infty B_{2,1}^{\frac{3}{4}}} \lesssim \|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B_{2,1}^{\frac{7}{4}}}^2 e^{CV_\varepsilon(T)}.$$

Plugging this inequality into (10), we obtain

$$\|\Gamma_\varepsilon\|_{L_T^4 L^\infty} \lesssim \varepsilon^{\frac{1}{4}} \big(\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B_{2,1}^{\frac{3}{4}}} + T\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B_{2,1}^{\frac{7}{4}}}^2 e^{CV_\varepsilon(T)} \big)$$
$$\lesssim C_0 \varepsilon^{\frac{1}{4}} (1 + T) e^{CV_\varepsilon(T)}.$$

Remark that the real part of Γ_ε is the compressible part of the velocity v_ε, then

$$\|\mathbb{Q}v_\varepsilon\|_{L_T^4 L^\infty} \lesssim C_0 \varepsilon^{\frac{1}{4}} (1 + T) e^{CV_\varepsilon(T)}.$$

We can prove in the similar way that,

$$\|\Upsilon_\varepsilon\|_{L_T^4 L^\infty} \lesssim C_0 \varepsilon^{\frac{1}{4}} (1 + T) e^{CV_\varepsilon(T)}$$

and therefore

$$\|c_\varepsilon\|_{L_T^4 L^\infty} \lesssim C_0 \varepsilon^{\frac{1}{4}} (1 + T) e^{CV_\varepsilon(T)}.$$

The proof of the Proposition 2 is now achieved. □

4. LOGARITHMIC ESTIMATE

The purpose of this paragraph is to study the linear growth of the norm $B_{\infty,1}^0$ for the following compressible transport model,

(11)
$$\begin{cases} \partial_t f + v \cdot \nabla f + f \operatorname{div} v = 0 \\ f_{|t=0} = f_0. \end{cases}$$

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We point out that the dynamics of the vorticity for the system (3) is governed by an equation of type (11). In the framework of the incompressible vector fields, that is div $v = 0$, Vishik [28] established the following linear growth for Besov regularity with zero index,

$$\|f(t)\|_{B^0_{\infty,1}} \leq C\|f_0\|_{B^0_{\infty,1}}\left(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\right).$$

A simple proof for this result was given in [13] and where we extended the Vishik's result to a transport-diffusion model. Our main goal here is to valid the previous linear growth to the compressible model and this is very crucial for the proof of the Theorem 1. Our result reads as follows,

Theorem 2. *Let v be a smooth vector field and f be a smooth solution of the transport equation* (11). *Then for every $1 \leq p < +\infty$, there exists a constant C depending only on p such that*

$$\|f(t)\|_{B^0_{\infty,1}} \leq C\|f_0\|_{B^0_{\infty,1}}\left(1 + e^{C\|\nabla v\|_{L^1_t L^\infty}}\|\mathrm{div}\, v\|^2_{L^1_t B^{\frac{2}{p}}_{p,1}}\right)\left(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\right).$$

Proof. First we observe that the above estimate reduces in the incompressible case to the Vishik's estimate. Here, we will try to use the same approach of [13], but as we will see the lack of the incompressibility brings more technical difficulties. Let us denote by ψ the flow associated to the velocity v :

$$\psi(t,x) = x + \int_0^t v\big(\tau, \psi(\tau,x)\big)d\tau, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2.$$

We will make use of the following estimate for the flow and its inverse ψ^{-1}

$$(12) \qquad \left\|\nabla\big(\psi(\tau, \psi^{-1}(t,\cdot))\big)\right\|_{L^\infty} \leq e^{\left|\int_\tau^t \|\nabla v(s)\|_{L^\infty} ds\right|}.$$

We set $g(t,x) = f(t, \psi(t,x))$ then it is clear that g satisfies the equation

$$\partial_t g(t,x) + (\mathrm{div}\, v)(t, \psi(t,x))g(t,x) = 0.$$

Therefore we obtain

$$g(t,x) = f_0(x)e^{-\int_0^t (\mathrm{div}\, v)(\tau, \psi(\tau,x))d\tau}.$$

It follows that

$$f(t,x) = f_0(\psi^{-1}(t,x))e^{-\int_0^t (\mathrm{div}\, v)(\tau, \psi(\tau, \psi^{-1}(t,x)))d\tau}.$$

Set $h(x) = e^x - 1$ and $W(t,x) = -\int_0^t (\mathrm{div}\, v)(\tau, \psi(\tau, \psi^{-1}(t,x)))d\tau$, then we infer

$$f(t,x) = f_0(\psi^{-1}(t,x))\big(h(W(t,x)) + 1\big).$$

Thus the problem reduces to the establishment of a composition law in $B^0_{\infty,1}$ space. For this purpose, we will use the following law product which can be obtained by using Bony's decomposition [5]: for $1 \leq p < \infty$,

$$\|uv\|_{B^0_{\infty,1}} \leq C\|u\|_{B^{\frac{2}{p}}_{p,1}}\|v\|_{B^0_{\infty,1}}.$$

Therefore

$$(13) \qquad \|f(t)\|_{B^0_{\infty,1}} \leq C\|f_0(\psi^{-1}(t,\cdot))\|_{B^0_{\infty,1}}\big(\|h(W(t,\cdot))\|_{B^{\frac{2}{p}}_{p,1}} + 1\big).$$

At this stage we need to the following result.

$$\|h(W(t,\cdot))\|_{B^{\frac{2}{p}}_{p,1}} \leq C\|W(t,\cdot)\|_{B^{\frac{2}{p}}_{p,1}} e^{C\|W(t)\|_{L^\infty}}.$$

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The proof of this estimate can be done as follows. By definition we get easily

$$\|h(W(t,\cdot))\|_{B_{p,1}^{\frac{2}{p}}} \le \sum_{n\ge 1} \frac{1}{n!} \|W^n(t,\cdot)\|_{B_{p,1}^{\frac{2}{p}}}.$$

According to the law product

$$\|u^2\|_{B_{p,1}^{\frac{2}{p}}} \le C\|u\|_{L^\infty}\|u\|_{B_{p,1}^{\frac{2}{p}}}, p < +\infty,$$

and using the induction principle we infer that for $n \ge 1$,

$$\|W^n(t,\cdot)\|_{B_{p,1}^{\frac{2}{p}}} \le C^{n-1}\|W(t,\cdot)\|_{L^\infty}^{n-1}\|W(t,\cdot)\|_{B_{p,1}^{\frac{2}{p}}}.$$

Therefore

$$\begin{aligned}
\|h(W(t,\cdot))\|_{B_{p,1}^{\frac{2}{p}}} &\le \|W\|_{B_{p,1}^{\frac{2}{p}}} \sum_{n\ge 1} \frac{C^{n-1}}{n!}\|W\|_{L^\infty}^{n-1}\\
&\le \|W\|_{B_{p,1}^{\frac{2}{p}}} \sum_{n\ge 0} C^n \frac{\|W\|_{L^\infty}^n}{(n+1)!}\\
&\le \|W\|_{B_{p,1}^{\frac{2}{p}}} \sum_{n\ge 0} C^n \frac{\|W\|_{L^\infty}^n}{n!}\\
&\le \|W\|_{B_{p,1}^{\frac{2}{p}}} e^{C\|W\|_{L^\infty}}.
\end{aligned}$$

This ends the proof of the desired inequality. Thus it follows that

$$\begin{aligned}
\|h(W(t,\cdot))\|_{B_{p,1}^{\frac{2}{p}}} &\lesssim \|W(t,\cdot)\|_{B_{p,1}^{\frac{2}{p}}} e^{C\|W(t)\|_{L^\infty}}\\
&\lesssim \|W(t,\cdot)\|_{B_{p,1}^{\frac{2}{p}}} e^{C\int_0^t \|\operatorname{div} v(\tau)\|_{L^\infty} d\tau}.
\end{aligned}$$

(14)

To estimate $W(t)$ we set $k_\tau(t,x) = \operatorname{div} v(\tau,\psi(\tau,\psi^{-1}(t,x)))$, then $k_\tau(t,\psi(t,x)) = \operatorname{div} v(\tau,\psi(\tau,x))$ and it follows that

$$\begin{cases} \partial_t k_\tau + v\cdot\nabla k_\tau = 0 \\ k_\tau(\tau,x) = \operatorname{div} v(\tau,x). \end{cases}$$

It remains to estimate $\|k_\tau(t)\|_{B_{p,1}^{\frac{2}{p}}}$. For this aim, we use Lemma 3-(1), yielding for $p > 2$ to

$$\|k_\tau(t)\|_{B_{p,1}^{\frac{2}{p}}} \le C\|\operatorname{div} v(\tau)\|_{B_{p,1}^{\frac{2}{p}}} e^{|\int_\tau^t \|\nabla v(s)\|_{L^\infty} ds|}.$$

Combining the definition of W with this latter estimate we get

$$\begin{aligned}
\|W(t)\|_{B_{p,1}^{\frac{2}{p}}} &\le \int_0^t \|k_\tau(t)\|_{B_{p,1}^{\frac{2}{p}}} d\tau\\
&\le C\int_0^t \|\operatorname{div} v(\tau)\|_{B_{p,1}^{\frac{2}{p}}} e^{\int_\tau^t \|\nabla v(s)\|_{L^\infty} ds} d\tau.
\end{aligned}$$

Plugging this estimate into (14) we find

$$\|h(W(t,\cdot))\|_{B_{p,1}^{\frac{2}{p}}} \le Ce^{C\|\nabla v\|_{L_t^1 L^\infty}} \int_0^t \|\operatorname{div} v(\tau)\|_{B_{p,1}^{\frac{2}{p}}} d\tau.$$

Inserting this estimate in (13) we obtain

$$(15) \qquad \|f(t)\|_{B^0_{\infty,1}} \leq C\|f_0(\psi^{-1}(t,\cdot))\|_{B^0_{\infty,1}} \left(1 + e^{C\|\nabla v\|_{L^1_t L^\infty}} \int_0^t \|\operatorname{div} v(\tau)\|_{B^{\frac{2}{p}}_{p,1}} d\tau\right)$$

It remains to estimate $\|f_0(\psi^{-1}(t,\cdot))\|_{B^0_{\infty,1}}$. Set $w(t,x) = f_0(\psi^{-1}(t,x))$ then it satisfies the transport equation

$$\begin{cases} \partial_t w + v \cdot \nabla w = 0 \\ w(0,x) = f_0(x). \end{cases}$$

We will split the initial data into Fourier modes $f_0 = \sum\limits_{q \geq -1} \Delta_q f_0$ and for each frequency $q \geq -1$, we denote by $\tilde{w}_q$ the unique global solution of the initial value problem

$$(16) \qquad \begin{cases} \partial_t \tilde{w}_q + v \cdot \nabla \tilde{w}_q = 0 \\ \tilde{w}_q(0,\cdot) = \Delta_q f_0. \end{cases}$$

According to Lemma 3-(2), we have

$$\|\tilde{w}_q(t)\|_{B^{\pm s}_{\infty,\infty}} \leq C\|\Delta_q f_0\|_{B^{\pm s}_{\infty,\infty}} e^{CV(t)}, \quad \forall \, 0 < s < \min(1, \tfrac{2}{p}),$$

with $V(t) := \|\nabla v\|_{L^1_t L^\infty} + \|\operatorname{div} v\|_{L^1_t B^{\frac{2}{p}}_{p,\infty}}$. Combined with the definition of Besov spaces, this implies for $j, q \geq -1$

$$(17) \qquad \|\Delta_j \tilde{w}_q(t)\|_{L^\infty} \leq C 2^{-s|j-q|}\|\Delta_q f_0\|_{L^\infty} e^{CV(t)}.$$

By linearity and the uniqueness of the solution, it is obvious that

$$w = \sum_{q=-1}^{\infty} \tilde{w}_q.$$

From the definition of Besov spaces we have

$$(18) \qquad \|w(t)\|_{B^0_{\infty,1}} \leq \sum_{|j-q| \geq N} \|\Delta_j \tilde{w}_q(t)\|_{L^\infty} + \sum_{|j-q| < N} \|\Delta_j \tilde{w}_q(t)\|_{L^\infty},$$

where $N \in \mathbb{N}$ is to be chosen later. To deal with the first sum we use (17)

$$\sum_{|j-q| \geq N} \|\Delta_j \tilde{w}_q(t)\|_{L^\infty} \lesssim 2^{-Ns} \sum_{q \geq -1} \|\Delta_q f_0\|_{L^\infty} e^{CV(t)}$$

$$(19) \qquad\qquad\qquad\qquad\qquad \lesssim 2^{-Ns}\|f_0\|_{B^0_{\infty,1}} e^{CV(t)}.$$

For the second sum in the right-hand side of (18), we use the fact that the operator Δ_j maps L^∞ into itself uniformly with respect to j,

$$\sum_{|j-q| < N} \|\Delta_j \tilde{w}_q(t)\|_{L^\infty} \lesssim \sum_{|j-q| < N} \|\tilde{w}_q(t)\|_{L^\infty}.$$

Applying the maximum principle to the system (16) yields

$$\|\tilde{w}_q(t)\|_{L^\infty} \leq \|\Delta_q f_0\|_{L^\infty}.$$

So, it holds that

$$\sum_{|j-q| < N} \|\Delta_j \tilde{w}_q(t)\|_{L^\infty} \lesssim N\|f_0\|_{B^0_{\infty,1}}.$$

The outcome is the following

$$\|w(t)\|_{B^0_{\infty,1}} \lesssim \|f_0\|_{B^0_{\infty,1}} \left(2^{-Ns} e^{CV(t)} + N\right).$$

Choosing N such that

$$N = \left[\frac{CV(t)}{s \log 2} + 1\right],$$

we get

$$\|w(t)\|_{B^0_{\infty,1}} \leq C\|f_0\|_{B^0_{\infty,1}} \left(1 + \int_0^t \left(\|\nabla v(\tau)\|_{L^\infty} + \|\mathrm{div}\, v(\tau)\|_{B^{\frac{2}{p}}_{p,\infty}}\right) d\tau\right).$$

Therefore

$$\|f_0\big(\psi^{-1}(t,\cdot)\big)\|_{B^0_{\infty,1}} \leq C\|f_0\|_{B^0_{\infty,1}} \left(1 + \int_0^t \left(\|\nabla v(\tau)\|_{L^\infty} + \|\mathrm{div}\, v(\tau)\|_{B^{\frac{2}{p}}_{p,\infty}}\right) d\tau\right)$$

Inserting this estimate into (15) gives

$$\|f(t)\|_{B^0_{\infty,1}} \leq C\|f_0\|_{B^0_{\infty,1}} \left(1 + e^{C\|\nabla v\|_{L^1_t L^\infty}} \|\mathrm{div}\, v\|^2_{L^1_t B^{\frac{2}{p}}_{p,1}}\right)\left(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau\right).$$

This is the desired result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

5. Proofs of the main results

In this section we will firstly extend the result of Theorem 1 to the framework of the heterogeneous Besov spaces $B^{s,\Psi}_{2,1}$. Secondly we will see how to derive the proof of the Theorem 1 and ultimately we will devote the rest of the paper to the discussion of the proof of the general statement given in Theorem 3.

5.1. General statement. We will give a generalization of the Theorem 1.

Theorem 3. *Let $\Psi \in \mathcal{U}_\infty$ and $\{(v_{0,\varepsilon}, c_{0,\varepsilon})\}_{0<\varepsilon\leq 1}$ be a bounded family in $B^{2,\Psi}_{2,1}$ of initial data, that is*

$$\sup_{0<\varepsilon\leq 1} \|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B^{2,\Psi}_{2,1}} := M_0 < +\infty.$$

Then the system (3) admits a unique solution $(v_\varepsilon, c_\varepsilon) \in \mathcal{C}([0, T_\varepsilon]; B^{2,\Psi}_{2,1})$, with

$$T_\varepsilon = C_0 \log\log\{\Psi(\log(\varepsilon^{-1}))\}.$$

Moreover, there exists $\eta > 0$ such that for small ε and for all $0 \leq T \leq T_\varepsilon$,

$$\|(\mathrm{div}\, v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_T L^\infty} \leq C_0 \Psi^{-\eta}(\varepsilon), \qquad \|\omega_\varepsilon(t)\|_{L^\infty} \leq C_0,$$

$$\|\nabla v_\varepsilon\|_{L^1_T L^\infty} \leq C_0 e^{C_0 T} \quad \text{and} \quad \|(v_\varepsilon, c_\varepsilon)(T)\|_{B^{2,\Psi}_{2,1}} \lesssim C_0 e^{e^{C_0 T}}.$$

The constant C_0 may depend on M_0 and on the profile Ψ.
Assume in addition that the incompressible parts $(\mathbb{P}v_{0,\varepsilon})_{0<\varepsilon\leq 1}$ converge in L^2 to some v_0. Then the incompressible parts of the solution converge strongly to the global solution of the system (4) with initial data v_0. More precisely, for all $T > 0$ and for all $\widetilde{\Psi} \in \mathcal{U}$ such that

$$\lim_{q\to+\infty} \frac{\widetilde{\Psi}(q)}{\Psi(q)} = 0$$

we have

$$\lim_{\varepsilon\to 0} \|\mathbb{P}v_\varepsilon - v\|_{L^\infty_T B^{2,\widetilde{\Psi}}_{2,1}} = 0.$$

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Remark 3.   (1) We observe that the result for the subcritical regularities is a special case of the above theorem. Indeed, let $\{(v_{0,\varepsilon}, c_{0,\varepsilon})\}_{0<\varepsilon\leq 1}$ be a bounded family in $H^s, s > 2$. It is plain that the following embedding holds true: $H^s \hookrightarrow B_{2,1}^{2,\Psi}$, with $\Psi(q) = 2^{q\alpha}$ and $0 < \alpha < s - 2$. Thus we get by applying Theorem 3 that

$$T_\varepsilon = C_0 \log\log\{\frac{1}{\varepsilon}\}.$$

This result is in agreement with known results about the lifespan for the subcritical regularities.

(2) If Ψ has a polynomial growth: $\Psi(q) = (q + 2)^\alpha, \alpha > 0$ then it is easily seen that $\Psi \in \mathcal{U}_\infty$ and

$$T_\varepsilon = C_0 \log\log\log\{\frac{1}{\varepsilon}\}.$$

Before going further into the details of the proof of the Theorem 3, we will first show how to deduce the result of the Theorem 1.

5.2. **Proof of Theorem 1.** Since

$$\sum_{q\geq -1} 2^{2q} \sup_{0<\varepsilon\leq 1} \|(\Delta_q v_{0,\varepsilon}, \Delta_q c_{0,\varepsilon})\|_{L^2} < +\infty,$$

we deduce from Corollary 1 the existence of a profile $\Psi \in \mathcal{U}_\infty$ such that the family $(v_{0,\varepsilon}, c_{0,\varepsilon})_{0<\varepsilon\leq 1}$ is uniformly bounded in $B_{2,1}^{2,\Psi}$. Then according to Theorem 3, there exists a unique solution $(v_\varepsilon, c_\varepsilon)$ belonging to the space $\mathcal{C}([0, T_\varepsilon]; B_{2,1}^{2,\Psi})$ with

$$T_\varepsilon = C_0 \log\log\{\Psi(\log(\varepsilon^{-1}))\}.$$

Therefore it is clear that $(v_\varepsilon, c_\varepsilon) \in \mathcal{C}([0, T_\varepsilon]; B_{2,1}^2)$ and

$$\lim_{\varepsilon\to 0} T_\varepsilon = +\infty.$$

For the incompressible limit, we can apply the second part of Theorem 3 with the profile $\widetilde{\Psi}$ equal to a nonnegative constant and thus the space $B_{2,1}^{2,\widetilde{\Psi}}$ reduces to the usual Besov space $B_{2,1}^2$.
We will now give the complete proof of Theorem 3, which will be done in several steps. We start first with estimating the lifespan of the solutions and we discuss at the end the incompressible limit problem.

5.3. **Lifespan of the solutions.** We will give an a priori bound of T_ε and show that the acoustic parts vanish when the Mach number goes to zero.

Proof. Using Lemma 4, we get

$$\|\nabla v_\varepsilon(t)\|_{L^\infty} \lesssim \|v_\varepsilon(t)\|_{L^2} + \|\operatorname{div} v_\varepsilon(t)\|_{B_{\infty,1}^0} + \|\omega_\varepsilon(t)\|_{B_{\infty,1}^0}.$$

Integrating in time and using Proposition 1 we obtain

$$\begin{aligned}
\|\nabla v_\varepsilon\|_{L_T^1 L^\infty} &\lesssim \|v_\varepsilon\|_{L_T^1 L^2} + \|\operatorname{div} v_\varepsilon\|_{L_T^1 B_{\infty,1}^0} + \|\omega_\varepsilon\|_{L_T^1 B_{\infty,1}^0} \\
&\lesssim C_0 T e^{C\|\operatorname{div} v_\varepsilon\|_{L_T^1 L^\infty}} + \|\operatorname{div} v_\varepsilon\|_{L_T^1 B_{\infty,1}^0} + \|\omega_\varepsilon\|_{L_T^1 B_{\infty,1}^0} \\
&\lesssim C_0(1+T)e^{C\|\operatorname{div} v_\varepsilon\|_{L_T^1 B_{\infty,1}^0}} + \|\omega_\varepsilon\|_{L_T^1 B_{\infty,1}^0}.
\end{aligned}$$

Now we recall the following notation,

$$V_\varepsilon(T) = \|\nabla v_\varepsilon\|_{L_T^1 L^\infty} + \|\nabla c_\varepsilon\|_{L_T^1 L^\infty}.$$

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Then it follows that

$$(20) \qquad V_\varepsilon(T) \ \le \ C_0(1+T)e^{C\|(\operatorname{div} v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_T B^0_{\infty,1}}} + \|\omega_\varepsilon\|_{L^1_T B^0_{\infty,1}}.$$

Applying Theorem 2 gives

$$\|\omega_\varepsilon(t)\|_{B^0_{\infty,1}} \ \le \ C\|\omega_\varepsilon^0\|_{B^0_{\infty,1}}\left(1 + e^{C\|\nabla v_\varepsilon\|_{L^1_t L^\infty}}\|\operatorname{div} v_\varepsilon\|^2_{L^1_t B^{\frac{2}{p}}_{p,1}}\right)(1 + \|\nabla v_\varepsilon\|_{L^1_t L^\infty})$$

$$(21) \qquad \le \ C_0\left(1 + e^{CV_\varepsilon(t)}\|\operatorname{div} v_\varepsilon\|^2_{L^1_t B^{\frac{2}{p}}_{p,1}}\right)(1 + V_\varepsilon(t)).$$

To estimate $\|\operatorname{div} v_\varepsilon\|_{B^{\frac{2}{p}}_{p,1}}$, we use the following interpolation inequality: for $2 < p < \infty$,

$$\|\operatorname{div} v_\varepsilon\|_{B^{\frac{2}{p}}_{p,1}} \ \lesssim \ \|\operatorname{div} v_\varepsilon\|^{\frac{2}{p}}_{B^1_{2,1}}\|\operatorname{div} v_\varepsilon\|^{1-\frac{2}{p}}_{B^0_{\infty,1}}.$$

Integrating in time and using Bernstein and Hölder inequalities, we obtain

$$\|\operatorname{div} v_\varepsilon\|_{L^1_T B^{\frac{2}{p}}_{p,1}} \ \lesssim \ T^{\frac{2}{p}}\|\operatorname{div} v_\varepsilon\|^{\frac{2}{p}}_{L^\infty_T B^1_{2,1}}\|\operatorname{div} v_\varepsilon\|^{1-\frac{2}{p}}_{L^1_T B^0_{\infty,1}}$$

$$\lesssim \ T^{\frac{2}{p}}\|v_\varepsilon\|^{\frac{2}{p}}_{L^\infty_T B^2_{2,1}}\|\operatorname{div} v_\varepsilon\|^{1-\frac{2}{p}}_{L^1_T B^0_{\infty,1}}.$$

Choose $p = 4$ and using Proposition 1-(2) yields to

$$(22) \qquad \|\operatorname{div} v_\varepsilon\|^2_{L^1_T B^{\frac{1}{2}}_{4,1}} \ \lesssim \ C_0 T e^{CV_\varepsilon(T)}\|\operatorname{div} v_\varepsilon\|_{L^1_T B^0_{\infty,1}}.$$

Putting together (21) and (22) and integrating in time we find

$$\|\omega_\varepsilon\|_{L^1_T B^0_{\infty,1}} \lesssim C_0 \int_0^T \left(1 + te^{CV_\varepsilon(t)}\|\operatorname{div} v_\varepsilon\|_{L^1_t B^0_{\infty,1}}\right)(1 + V_\varepsilon(t))dt.$$

Inserting this estimate into (20)

$$(23) \quad V_\varepsilon(T) \le C_0(1+T)e^{C\|(\operatorname{div} v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_T B^0_{\infty,1}}} + C_0\int_0^T \left(1 + te^{CV_\varepsilon(t)}\|\operatorname{div} v_\varepsilon\|_{L^1_t B^0_{\infty,1}}\right)(1 + V_\varepsilon(t))dt.$$

It remains to estimate $\|(\operatorname{div} v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_T B^0_{\infty,1}}$. For this purpose we will develop an interpolation procedure between the Strichartz estimates for lower frequencies and the energy estimates for higher frequencies. More precisely, let $N \in \mathbb{N}^*$ that will be judiciously fixed later. Then using Bernstein inequality combined with the continuity of the operator $\mathbb{Q}$ on the Lebesgue space L^2, we find

$$\|\operatorname{div} v_\varepsilon\|_{B^0_{\infty,1}} \ = \ \|\operatorname{div} \mathbb{Q} v_\varepsilon\|_{B^0_{\infty,1}}$$

$$\le \ \sum_{-1 \le q \le N}\|\Delta_q \operatorname{div} \mathbb{Q} v_\varepsilon\|_{L^\infty} + \sum_{q > N}\|\Delta_q \operatorname{div} \mathbb{Q} v_\varepsilon\|_{L^\infty}$$

$$\lesssim \ \sum_{q \le N}2^q\|\Delta_q \mathbb{Q} v_\varepsilon\|_{L^\infty} + \frac{1}{\Psi(N)}\sum_{q > N}\Psi(q)2^{2q}\|\Delta_q \mathbb{Q} v_\varepsilon\|_{L^2}$$

$$\lesssim \ 2^N\|\mathbb{Q} v_\varepsilon\|_{L^\infty} + \frac{1}{\Psi(N)}\|v_\varepsilon\|_{B^{2,\Psi}_{2,1}}.$$

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We have used the fact that the profile Ψ is a nondecreasing function. Now integrating in time and using Proposition 2 and Proposition 1-(2)

$$\|\operatorname{div} v_\varepsilon\|_{L^1_T B^0_{\infty,1}} \lesssim 2^N T^{\frac{3}{4}} \|\mathbb{Q} v_\varepsilon\|_{L^4_T L^\infty} + \frac{T}{\Psi(N)} \|v_\varepsilon\|_{L^\infty_T B^{2,\Psi}_{2,1}}$$

$$\leq C_0\big(T^{\frac{3}{4}}(1+T)+T\big)e^{CV_\varepsilon(T)}\Big(\varepsilon^{\frac{1}{4}} 2^N + \frac{1}{\Psi(N)}\Big)$$

$$\leq C_0(1+T^{\frac{7}{4}})e^{CV_\varepsilon(T)}\Big(\varepsilon^{\frac{1}{4}} 2^N + \frac{1}{\Psi(N)}\Big).$$

By similar computations we get

$$\|\nabla c_\varepsilon\|_{L^1_T B^0_{\infty,1}} \leq C_0(1+T^{\frac{7}{4}})e^{CV_\varepsilon(T)}\Big(\varepsilon^{\frac{1}{4}} 2^N + \frac{1}{\Psi(N)}\Big).$$

Therefore

$$\|(\operatorname{div} v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_T B^0_{\infty,1}} \leq C_0(1+T^{\frac{7}{4}})e^{CV_\varepsilon(T)}\Big(\varepsilon^{\frac{1}{4}} 2^N + \frac{1}{\Psi(N)}\Big).$$

Now, we choose N such that,

$$N \approx \log\Big(\frac{1}{\varepsilon^{\frac{1}{8}}}\Big).$$

Hence we get

$$\|(\operatorname{div} v_\varepsilon, \nabla c_\varepsilon\|_{L^1_T B^0_{\infty,1}} \leq C_0(1+T^{\frac{7}{4}})e^{CV_\varepsilon(T)}\Big(\varepsilon^{\frac{1}{8}} + \frac{1}{\Psi\big(\log(\frac{1}{\varepsilon^{\frac{1}{8}}})\big)}\Big).$$

According to the Remark 2, the profile Ψ has at most an exponential growth: there exists $\alpha > 0$ such that

$$\forall x \geq -1; \quad \Psi(x) \leq Ce^{\alpha x}.$$

Hence we get

(24)
$$\varepsilon^{\frac{1}{8}} \leq \frac{C^{\frac{1}{\alpha}}}{\Psi^{\frac{1}{\alpha}}(\log(\frac{1}{\varepsilon^{\frac{1}{8}}})}.$$

Let $\beta = \min(1, \frac{1}{\alpha})$ then

(25)
$$\|(\operatorname{div} v_\varepsilon, \nabla c_\varepsilon\|_{L^1_T B^0_{\infty,1}} \leq C_0(1+T^{\frac{7}{4}})e^{CV_\varepsilon(T)}\Phi(\varepsilon),$$

with

$$\Phi(\varepsilon) := \frac{1}{\Psi^\beta\big(\log(\frac{1}{\varepsilon^{\frac{1}{8}}})\big)}.$$

Plugging (25) into (23) and using Gronwall's inequality we obtain

$$V_\varepsilon(T) \leq C_0(1+T)e^{C_0(1+T^{\frac{7}{4}})\Phi(\varepsilon)e^{CV_\varepsilon(T)}} + C_0 \int_0^T \big(1+(1+t^{\frac{11}{4}})e^{CV_\varepsilon(t)}\Phi(\varepsilon)\big)(1+V_\varepsilon(t))dt$$

(26)
$$\leq C_0 e^{C_0 T} \exp\{C_0(1+T^{\frac{15}{4}})\Phi(\varepsilon)e^{CV_\varepsilon(T)}\}.$$

We choose T_ε such that,

(27)
$$e^{e^{C_0 T_\varepsilon}} = \Phi^{-\frac{1}{2}}(\varepsilon),$$

then we claim that for small ε and $0 \leq t \leq T_\varepsilon$ we have

(28)
$$e^{CV_\varepsilon(t)} \leq \Phi^{-\frac{2}{3}}(\varepsilon).$$

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Indeed, define
$$J_\varepsilon := \left\{ t \in [0, T_\varepsilon]; e^{CV_\varepsilon(t)} \le \Phi^{-\frac{2}{3}}(\varepsilon) \right\}.$$
This set is nonempty since $0 \in J_\varepsilon$. It is also closed by the continuity of the mapping $t \mapsto V_\varepsilon(t)$. It remains to prove that J_ε is an open subset of $[0, T_\varepsilon]$ and thus we deduce $J_\varepsilon = [0, T_\varepsilon]$. Let $t \in J_\varepsilon$, then using (26) we get for small ε

(29)
$$e^{CV_\varepsilon(t)} \le C_0 e^{\exp\{C_0 T_\varepsilon + C_0 T_\varepsilon^{\frac{15}{4}} \Phi^{\frac{1}{3}}(\varepsilon)\}}.$$

From (27), we infer
$$C_0 T_\varepsilon = \log\log \Phi^{-\frac{1}{2}}(\varepsilon) \approx \log\log \Psi\left(\log(\frac{1}{\varepsilon^{\frac{1}{8}}})\right).$$

Since
$$\lim_{\varepsilon \to 0} \Phi^{\frac{1}{3}}(\varepsilon) \left\{ \log\log \Phi^{-\frac{1}{2}}(\varepsilon) \right\}^{\frac{15}{4}} = 0,$$
then for sufficiently small ε and for $t \in [0, T_\varepsilon]$, we get
$$\begin{aligned} e^{CV_\varepsilon(t)} &\le 2 C_0 \Phi^{-\frac{1}{2}}(\varepsilon) \\ &< \Phi^{-\frac{2}{3}}(\varepsilon). \end{aligned}$$

This proves that t belongs to the interior of J_ε and consequently J_ε is an open subset of $[0, T_\varepsilon]$. Finally we get $J_\varepsilon = [0, T_\varepsilon]$. From the previous estimate, we obtain for all $T \in [0, T_\varepsilon]$

(30)
$$(1 + T^{\frac{15}{4}}) e^{CV_\varepsilon(T)} \Phi(\varepsilon) \lesssim \left\{ \log\log \Phi^{-\frac{1}{2}}(\varepsilon) \right\}^{\frac{15}{4}} \Phi^{\frac{1}{3}}(\varepsilon) \lesssim 1.$$

Inserting (30) into (26), yields for $0 \le T \le T_\varepsilon$

(31)
$$V_\varepsilon(T) \le C_0 e^{C_0 T}.$$

In particular we have obtained
$$\|\nabla v_\varepsilon\|_{L_T^1 L^\infty} \le C_0 e^{C_0 T}.$$

Plugging the estimate (30) into (25), we get for small ε and for $T \in [0, T_\varepsilon]$ that
$$\begin{aligned} \|(\operatorname{div} v_\varepsilon, \nabla c_\varepsilon)\|_{L_T^1 B_{\infty,1}^0} &\le C_0 \left\{ \log\log \Phi^{-\frac{1}{2}}(\varepsilon) \right\}^{\frac{7}{4}} \Phi^{\frac{1}{3}}(\varepsilon) \\ \end{aligned}$$

(32)
$$\le C_0 \Phi^{\frac{1}{4}}(\varepsilon).$$

Using Proposition 2, (28) and (24) we obtain for small ε,
$$\begin{aligned} \|\mathbb{Q} v_\varepsilon\|_{L_T^4 L^\infty} + \|c_\varepsilon\|_{L_T^4 L^\infty} &\le C_0 \varepsilon^{\frac{1}{4}} (1 + T) e^{CV_\varepsilon(T)} \\ &\le C_0 \varepsilon^{\frac{1}{4}} \Phi^{-\frac{2}{3}}(\varepsilon) \log\log \Phi^{-\frac{1}{2}}(\varepsilon) \\ &\le C_0 \Phi^{\frac{4}{3}}(\varepsilon) \log\log \Phi^{-\frac{1}{2}}(\varepsilon) \\ \end{aligned}$$

(33)
$$\le C_0 \Phi(\varepsilon).$$

Let us now move to the estimate of the vorticity. First, recall that ω_ε satisfies the compressible transport equation
$$\partial_t \omega_\varepsilon + v_\varepsilon \cdot \nabla \omega_\varepsilon + \omega_\varepsilon \operatorname{div} v_\varepsilon = 0.$$
Consequently we get by using Gronwall's inequality that
$$\|\omega_\varepsilon(t)\|_{L^\infty} \le \|\omega_{0,\varepsilon}\|_{L^\infty} e^{\|\operatorname{div} v_\varepsilon\|_{L_t^1 L^\infty}}.$$
It suffices to apply (32), leading to the following estimate: for all $T \in [0, T_\varepsilon]$
$$\|\omega_\varepsilon(T)\|_{L^\infty} \le C_0.$$

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Finally, to estimate $\|(v_\varepsilon, c_\varepsilon)(T)\|_{B^{2,\Psi}_{2,1}}$ we use Proposition 1 combined with (31) to find,

$$\|(v_\varepsilon, c_\varepsilon)(T)\|_{B^{2,\Psi}_{2,1}} \leq C\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B^{2,\Psi}_{2,1}} e^{CV_\varepsilon(T)}$$

(34)
$$\leq C_0 e^{\exp\{C_0 T\}}.$$

We observe that in Theorem 3 the formula for the lifespan involves the quantity $\log\log \Psi(\log \varepsilon^{-1})$ instead of $\log\log \Psi(\log \varepsilon^{-\frac{1}{8}})$. This is due to the fact that for any given $\lambda > 0$ the function defined by $\Psi_\lambda(x) := \Psi(\lambda x)$ belongs to the same class $\mathcal{U}_\infty$.
This completes the proof of the first part of Theorem 3. $\qquad\square$

5.4. **Incompressible limit.** In this paragraph we will sketch the proof of the second part of Theorem 3 which deals with the incompressible limit.

Proof. We will proceed in two steps. In the first one, we will prove that for any fixed $T > 0$, the family $(\mathbb{P}v_\varepsilon)_\varepsilon$ converges strongly in $L^\infty_T L^2$, when $\varepsilon \to 0$, to the solution v of the incompressible Euler equations with initial data v_0. It is worthy pointing out that similarly to the Remark 1-(2) we can show that the limit v_0 belongs also to the same space $B^{2,\Psi}_{2,1}$. In the second step we will show how to get the strong convergence in the Besov spaces $B^{2,\widetilde{\Psi}}_{2,1}$. We mention that the limit system (4) is globally well-posed if $v_0 \in B^{2,\Psi}_{2,1}$. Indeed, combining the embedding $B^{2,\Psi}_{2,1} \hookrightarrow B^2_{2,1}$ with the Vishik's result [28] we get that the system (4) admits a unique global solution such that $v \in \mathcal{C}(\mathbb{R}_+; B^2_{2,1})$, with the following Lipschitz bound

$$\|\nabla v(t)\|_{L^\infty} \leq C_0 e^{C_0 t}.$$

This latter a priori estimate together with Proposition 1, which remains valid for (4), gives the global persistence of the initial regularity $B^{2,\Psi}_{2,1}$. More precisely, we get

(35)
$$\|v(t)\|_{B^{2,\Psi}_{2,1}} \leq C_0 e^{\exp C_0 t}.$$

Let $\varepsilon > 0$ and set

$$\zeta_\varepsilon := v_\varepsilon - v, \quad w_\varepsilon = \mathbb{P}v_\varepsilon - v.$$

By applying Leray's projector $\mathbb{P}$ to the first equation of (3), we obtain

$$\partial_t \mathbb{P}v_\varepsilon + \mathbb{P}(v_\varepsilon \cdot \nabla v_\varepsilon) = 0.$$

Taking the difference between the above equation and (4) yields to

$$\partial_t w_\varepsilon + \mathbb{P}(v_\varepsilon \cdot \nabla \zeta_\varepsilon) + \mathbb{P}(\zeta_\varepsilon \cdot \nabla v) = 0.$$

We take the L^2 inner product of the above equation with w_ε, integrating by parts and using the identities

$$\zeta_\varepsilon = w_\varepsilon + \mathbb{Q}v_\varepsilon, \quad \mathbb{P}w_\varepsilon = w_\varepsilon,$$

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we get,

$$\frac{1}{2}\frac{d}{dt}\| w_\varepsilon(t)\|_{L^2}^2 = -\int_{\mathbb{R}^2}(v_\varepsilon\cdot\nabla\zeta_\varepsilon)\mathbb{P}\, w_\varepsilon dx - \int_{\mathbb{R}^2}(\zeta_\varepsilon\cdot\nabla v)\mathbb{P}\, w_\varepsilon dx$$

$$= -\int_{\mathbb{R}^2}\big(v_\varepsilon\cdot\nabla(\, w_\varepsilon + \mathbb{Q}v_\varepsilon)\big)\, w_\varepsilon dx - \int_{\mathbb{R}^2}\big((\, w_\varepsilon + \mathbb{Q}v_\varepsilon)\cdot\nabla v\big)\, w_\varepsilon dx$$

$$= \frac{1}{2}\int_{\mathbb{R}^2}|\, w_\varepsilon|^2\mathrm{div}\, v_\varepsilon dx - \int_{\mathbb{R}^2}(\, w_\varepsilon\cdot\nabla v)\, w_\varepsilon dx$$

$$\quad - \int_{\mathbb{R}^2}(v_\varepsilon\cdot\nabla\mathbb{Q}v_\varepsilon + \mathbb{Q}v_\varepsilon\cdot\nabla v)\, w_\varepsilon dx$$

$$\leq \Big(\frac{1}{2}\|\mathrm{div}\, v_\varepsilon(t)\|_{L^\infty} + \|\nabla v(t)\|_{L^\infty}\Big)\|\, w_\varepsilon(t)\|_{L^2}^2$$

$$\quad + \Big(\|v_\varepsilon(t)\|_{L^2}\|\nabla\mathbb{Q}v_\varepsilon(t)\|_{L^\infty} + \|\mathbb{Q}v_\varepsilon(t)\|_{L^\infty}\|\nabla v(t)\|_{L^2}\Big)\|\, w_\varepsilon(t)\|_{L^2}.$$

This implies that

$$\frac{d}{dt}\|\, w_\varepsilon(t)\|_{L^2} \leq \Big(\frac{1}{2}\|\mathrm{div}\, v_\varepsilon(t)\|_{L^\infty} + \|\nabla v(t)\|_{L^\infty}\Big)\|\, w_\varepsilon(t)\|_{L^2}$$

$$\quad + \|v_\varepsilon(t)\|_{L^2}\|\nabla\mathbb{Q}v_\varepsilon(t)\|_{L^\infty} + \|\mathbb{Q}v_\varepsilon(t)\|_{L^\infty}\|\nabla v(t)\|_{L^2}.$$

Integrating in time and using Gronwall's inequality yield to

(36) $$\|\, w_\varepsilon(t)\|_{L^2} \leq \big(\|w_\varepsilon^0\|_{L^2} + F_\varepsilon(t)\big)e^{\frac{1}{2}\|\mathrm{div}\, v_\varepsilon\|_{L^1_t L^\infty} + \|\nabla v\|_{L^1_t L^\infty}},$$

where

$$F_\varepsilon(t) := \|v_\varepsilon\|_{L^\infty_t L^2}\|\nabla\mathbb{Q}v_\varepsilon(t)\|_{L^1_t L^\infty} + \|\mathbb{Q}v_\varepsilon\|_{L^1_t L^\infty}\|\nabla v\|_{L^\infty_t L^2}$$

$$\lesssim \|v_\varepsilon\|_{L^\infty_t L^2}\|\nabla\mathbb{Q}v_\varepsilon\|_{L^1_t L^\infty} + \|\mathbb{Q}v_\varepsilon\|_{L^1_t L^\infty}\|\omega\|_{L^\infty_t L^2},$$

where ω denotes the vorticity of v. To estimate $\|\nabla\mathbb{Q}v_\varepsilon\|_{L^\infty}$ we use Bernstein inequality,

$$\|\nabla\mathbb{Q}v_\varepsilon\|_{L^\infty} \leq \|\Delta_{-1}\nabla\mathbb{Q}v_\varepsilon\|_{L^\infty} + \sum_{q\geq 0}\|\Delta_q\nabla\mathbb{Q}v_\varepsilon\|_{L^\infty}$$

$$\lesssim \|\mathbb{Q}v_\varepsilon\|_{L^\infty} + \sum_{q\geq 0}\|\Delta_q\mathrm{div}\, v_\varepsilon\|_{L^\infty}$$

$$\lesssim \|\mathbb{Q}v_\varepsilon\|_{L^\infty} + \|\mathrm{div}\, v_\varepsilon\|_{B^0_{\infty,1}}.$$

Integrating in time

$$\|\nabla\mathbb{Q}v_\varepsilon\|_{L^1_t L^\infty} \leq \|\mathbb{Q}v_\varepsilon\|_{L^1_t L^\infty} + \|\mathrm{div}\, v_\varepsilon\|_{L^1_t B^0_{\infty,1}}.$$

By virtue of (32) and (33) one has for small ε and $t\in[0,T]$,

$$\|\nabla\mathbb{Q}v_\varepsilon\|_{L^1_t L^\infty} + \|\mathbb{Q}v_\varepsilon\|_{L^1_t L^\infty} \leq C_0\Phi^{\frac{1}{4}}(\varepsilon).$$

On the other hand we get from Proposition 1-(1) and (32)

$$\|v_\varepsilon(t)\|_{L^\infty_t L^2} \leq C_0.$$

To estimate $\|\omega(t)\|_{L^2}$ we use the fact that the vorticity is transported for the incompressible flow and thus

$$\|\omega(t)\|_{L^2} = \|\omega_0\|_{L^2}.$$

Therefore we obtain for $t\in[0,T]$ and for small ε,

$$F_\varepsilon(t) \leq C_0\Phi^{\frac{1}{4}}(\varepsilon).$$

According to Vishik's result [28] we have for all $t \geq 0$

$$\|\nabla v(t)\|_{L^\infty} \leq C_0 e^{C_0 t}.$$

Inserting the previous estimates in (36) we find for all $t \in [0, T]$ and small values of ε,

(37)
$$\|w_\varepsilon(t)\|_{L^2} \leq C_0 e^{\exp C_0 t} \big(\|\mathbb{P}v_{0,\varepsilon} - v_0\|_{L^2} + \Phi^{\frac{1}{4}}(\varepsilon) \big).$$

This achieves the proof of the strong convergence in $L^\infty_{loc}(\mathbb{R}_+; L^2)$.

Let us now turn to the proof of the strong convergence in the space $L^\infty_T B_{2,1}^{2,\widetilde{\Psi}}$. We recall that $\widetilde{\Psi}$ is any element of the set $\mathcal{U}$, see Definition 1, and satisfying in addition the assumption

$$\lim_{q \to +\infty} \frac{\widetilde{\Psi}(q)}{\Psi(q)} = 0.$$

By using (34) and the continuity of Leray's projector on the spaces $B_{2,1}^{2,\Psi}$ we get for $t \in [0, T]$ and for small ε

(38)
$$\|\mathbb{P}v_\varepsilon(t)\|_{B_{2,1}^{2,\Psi}} \leq C_0 e^{\exp C_0 t}.$$

Now let $N \in \mathbb{N}^\star$ be an arbitrary number. Then by (37), (35) and (38) we obtain

$$
\begin{aligned}
\|w_\varepsilon(t)\|_{B_{2,1}^{2,\widetilde{\Psi}}} &= \sum_{q=-1}^{N} \widetilde{\Psi}(q) 2^{2q} \|\Delta_q w_\varepsilon(t)\|_{L^2} + \sum_{q>N} \frac{\widetilde{\Psi}(q)}{\Psi(q)} \Psi(q) 2^{2q} \|\Delta_q w_\varepsilon(t)\|_{L^2} \\
&\leq C_0 e^{\exp C_0 t} \big(\|\mathbb{P}v_{0,\varepsilon} - v_0\|_{L^2} + \Phi^{\frac{1}{4}}(\varepsilon) \big) \widetilde{\Psi}(N) 2^{2N} + \rho_N \|(\mathbb{P}v_\varepsilon, v)(t)\|_{B_{2,1}^{2,\Psi}} \\
&\leq C_0 e^{\exp C_0 t} \Big(\big(\|\mathbb{P}v_{0,\varepsilon} - v_0\|_{L^2} + \Phi^{\frac{1}{4}}(\varepsilon) \big) \widetilde{\Psi}(N) 2^{2N} + \rho_N \Big),
\end{aligned}
$$

with

$$\rho_N := \max_{q \geq N} \frac{\widetilde{\Psi}(q)}{\Psi(q)}.$$

We remark that the $(\rho_N)_N$ is a decreasing sequence to zero. Consequently,

$$\limsup_{\varepsilon \to 0^+} \|w_\varepsilon\|_{L^\infty_T B_{2,1}^{2,\widetilde{\Psi}}} \leq C_0 e^{\exp C_0 T} \rho_N.$$

Combining this estimate with $\lim_{N \to +\infty} \rho_N = 0$, we get

$$\lim_{\varepsilon \to 0^+} \|w_\varepsilon\|_{L^\infty_T B_{2,1}^{2,\widetilde{\Psi}}} = 0.$$

This completes the proof of the strong convergence. $\qquad\square$

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